Geometric Sequences and Sums

Sequence

A Sequence is a set of things (usually numbers) that are in order.

Geometric Sequences

In a Geometric Sequence each term is found by multiplying the previous term by a constant.

Example:

$$2, 4, 8, 16, 32, 64, 128, 256, ...$$

This sequence has a factor of 2 between each number.

Each term (except the first term) is found by multiplying the previous term by 2.

In General you could write a Geometric Sequence like this:

$$\{a, ar, ar^2, ar^3, ...\}$$

where:

- $a$ is the first term, and
- $r$ is the factor between the terms (called the "common ratio")

Example: $\{1, 2, 4, 8, ...\}$
The sequence starts at 1 and doubles each time, so
- \( a=1 \) (the first term)
- \( r=2 \) (the "common ratio" between terms is a doubling)

So we would get:

\[
\{a, ar, ar^2, ar^3, \ldots \} = \{1, 1\times2, 1\times2^2, 1\times2^3, \ldots \} = \{1, 2, 4, 8, \ldots \}
\]

But be careful, \( r \) should not be 0:
- When \( r=0 \), you get the sequence \( \{a,0,0,\ldots\} \) which is not geometric

The Rule

You can also calculate **any term** using the Rule:

\[
x_n = ar^{n-1}
\]

(We use "n-1" because \( ar^0 \) is for the 1st term)

Example:

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number.

The values of \( a \) and \( r \) are:
- \( a = 10 \) (the first term)
- \( r = 3 \) (the "common ratio")

The Rule for any term is:

\[
x_n = 10 \times 3^{(n-1)}
\]

So, the 4th term would be:

\[
x_4 = 10 \times 3^{(4-1)} = 10 \times 3^3 = 10 \times 27 = 270
\]

And the 10th term would be:
\[ x_{10} = 10 \times 3^{10-1} = 10 \times 3^9 = 10 \times 19683 = 196830 \]

A Geometric Sequence can also have **smaller and smaller** values:

**Example:**

4, 2, 1, 0.5, 0.25, ...

This sequence has a factor of 0.5 (a half) between each number.

Its Rule is \[x_n = 4 \times (0.5)^{n-1}\]

**Why “Geometric” Sequence?**

Because it is like increasing the dimensions in **geometry:**

- a line is 1-dimensional and has a length of \(r^1\)
- in 2 dimensions a square has an area of \(r^2\)
- in 3 dimensions a cube has volume \(r^3\)
- \(\ldots\) etc (yes you can have 4 and more dimensions in mathematics).

Geometric Sequences are sometimes called Geometric Progressions (G.P.'s)

**Summing a Geometric Series**

When you need to sum a Geometric Sequence, there is a handy formula.

To sum:

\[ a + ar + ar^2 + \ldots + ar^{(n-1)} \]

Each term is \(ar^k\), where \(k\) starts at 0 and goes up to \(n-1\)
Use this formula:

\[ \sum_{k=0}^{n-1} (ar^k) = a \left(1 - r^n \right) \left(1 - r \right) \]

- \(a\) is the first term
- \(r\) is the "common ratio" between terms
- \(n\) is the number of terms

What is that funny symbol? It is called **Sigma Notation**

\[ \sum \] (called Sigma) means "sum up"

And below and above it are shown the starting and ending values:

It says "Sum up \(n\) where \(n\) goes from 1 to 4. Answer=10"

The formula is easy to use ... just "plug in" the values of \(a\), \(r\) and \(n\)

Example: Sum the first 4 terms of

\[ 10, 30, 90, 270, 810, 2430, \ldots \]

This sequence has a factor of 3 between each number.

The values of \(a\), \(r\) and \(n\) are:

- \(a = 10\) (the first term)
- \(r = 3\) (the "common ratio")
- \(n = 4\) (we want to sum the first 4 terms)

So:

\[ \sum_{k=0}^{n-1} (ar^k) = a \left( \frac{1 - r^n}{1 - r} \right) \]

Becomes:

\[ \sum_{k=0}^{4-1} (10 \cdot 3^k) = 10 \left( \frac{1 - 3^4}{1 - 3} \right) = 400 \]

You could check it yourself:
10 + 30 + 90 + 270 = 400

And, yes, it was easier to just add them in this case, because there were only 4 terms. But imagine you had to sum up lots of terms, then the formula is better to use.

Using the Formula

Let's see the formula in action:

Example: Grains of Rice on a Chess Board

On our page Binary Digits we give an example of grains of rice on a chess board. The question is asked:

When you place rice on the chess board:

- 1 grain on the first square,
- 2 grains on the second square,
- 4 grains on the third and so on,
- ...

... doubling the grains of rice on each square ...

... how many grains of rice in total?

So we have:

- \( a = 1 \) (the first term)
- \( r = 2 \) (doubles each time)
- \( n = 64 \) (64 squares on a chess board)

So:

\[
\sum_{k=0}^{n-1} (ar^k) = a \left( \frac{1 - r^n}{1 - r} \right)
\]

Becomes:

\[
\sum_{k=0}^{63} (ar^k) = 2 \left( \frac{1 - 2^{64}}{1 - 2} \right)
\]

\[
= (1 - 2^{64}) / (-1) = 2^{64} - 1
\]

\[
= 18,446,744,073,709,551,615
\]

Which was exactly the result we got on the Binary Digits page (thank goodness!)
And another example, this time with \( r \) less than 1:

Example: Add up the first 10 terms of the Geometric Sequence that halves each time:

\[
\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \}
\]

The values of \( a \), \( r \) and \( n \) are:

- \( a = \frac{1}{2} \) (the first term)
- \( r = \frac{1}{2} \) (halves each time)
- \( n = 10 \) (10 terms to add)

So:

\[
\sum_{k=0}^{n-1} (ar^k) = a \left( \frac{1 - r^n}{1 - r} \right)
\]

Becomes:

\[
\sum_{k=0}^{10-1} \left( \frac{1}{2} \right)^k = \frac{1}{2} \left( \frac{1 - \left( \frac{1}{2} \right)^{10}}{1 - \frac{1}{2}} \right)
\]

\[
= \frac{1}{2} \left( \frac{1 - \frac{1}{1024}}{\frac{1}{2}} \right)
\]

\[
= 1 - \frac{1}{1024}
\]

\[
= 0.9990234375
\]

Very close to 1.

(Question: if we continue to increase \( n \), what would happen?)

Why Does the Formula Work?

I want to show you why the formula works, because we get to use an interesting "trick" which is worth knowing.

First, we will call the whole sum "\( S \)":

\[
S = a + ar + ar^2 + \ldots + ar^{(n-2)} + ar^{(n-1)}
\]

Next, multiply \( S \) by \( r \):

\[
S \cdot r = ar + ar^2 + ar^3 + \ldots + ar^{(n-1)} + ar^n
\]

Notice that \( S \) and \( S \cdot r \) are similar?

Now subtract them!
\[ S = a + ar + ar^2 + \ldots + ar^{n-1} \]
\[ S \cdot r = -ar - ar^2 - \ldots - ar^{n-1} - ar^n \]
\[ S - S \cdot r = a (1 - r^n) = ar^n \]

*Wow! All the terms in the middle neatly cancel out.*
(That is the neat trick I wanted to show you.)

By subtracting \( S \cdot r \) from \( S \) we get a simple result:

\[ S - S \cdot r = a - ar^n \]

Let's rearrange it to find \( S \):

Factor out \( S \) and \( a \):

\[ S(1-r) = a(1-r^n) \]

Divide by \((1-r)\): \[ S = a(1-r^n)/(1-r) \]

Which is our formula (ta-da!):

\[ \sum_{k=0}^{n-1} (ar^k) = a \left( \frac{1-r^n}{1-r} \right) \]

### Infinite Geometric Series

So what happens when \( n \) goes to infinity?

Well ... when \( r \) is less than 1, then \( r^n \) goes to zero and we get:

\[ \sum_{k=0}^{\infty} (ar^k) = a \left( \frac{1}{1-r} \right) \]

**NOTE:** this does not work when \( r \) is 1 or more (or less than -1):

\( r \) must be between (but not including) \(-1 \) and \( 1 \)

and \( r \) should not be \( 0 \) because you get the sequence \((a,0,0,\ldots)\) which is not geometric

Let's bring back our previous example, and see what happens:
Example: Add up ALL the terms of the Geometric Sequence that halves each time:

\{ 1/2, 1/4, 1/8, 1/16, ... \}

We have:
- \( a = 1/2 \) (the first term)
- \( r = 1/2 \) (halves each time)

And so:

\[
\sum_{k=0}^{\infty} \left( \frac{1}{2} \cdot \left( \frac{1}{2} \right)^k \right) = \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2}} \right)
\]

\[
= \frac{1}{2} \times 1 = \frac{1}{2} = 1
\]

Yes ... adding \((1/2) + (1/4) + (1/8) + ...\) equals **exactly** 1.

Don't believe me? Just look at this square:

By adding up \((1/2) + (1/4) + (1/8) + ...\)

... you end up with the whole thing!

Recurring Decimal

On another page we asked "**Does 0.999... equal 1?**", well, let us see if we can calculate it:

Example: Calculate 0.999...

We can write a recurring decimal as a sum like this:

\[
0.999... = 0.9 + 0.09 + 0.009 + ... = 0.9 \cdot 0.1^0 + 0.9 \cdot 0.1^1 + 0.9 \cdot 0.1^2 + ...
\]

\[
= \sum_{k=0}^{\infty} 0.9 \cdot 0.1^k
\]

And now we can use the formula:

\[
\sum_{k=0}^{\infty} 0.9 \times 0.1^k = 0.9 \left( \frac{1}{1 - 0.1} \right) = 0.9 \left( \frac{1}{0.9} \right) = 1
\]

Yes! 0.999... **does** equal 1.
So there you have it ... Geometric Sequences (and their sums) can do all sorts of amazing and powerful things.