Honours Advanced Algebra

Unit 2: Polynomial Functions
Factors, Zeros, and Roots: Oh My! Learning Task (Task 5)

Date: ___________________ Period: ___________

Mathematical Goals
- Know and apply the Remainder Theorem
- Know and apply the Rational Root Theorem
- Know and apply the Factor Theorem
- Know and apply the Fundamental Theorem of Algebra
- Identify zeros and factors of polynomial functions
- Classify roots as rational, irrational, real, and/or imaginary

Common Core State Standards

Interpret the structure of expressions.
MCC9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context. ★
MCC9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients. ★
MCC9-12.A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. ★
MCC9-12.A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Perform arithmetic operations on polynomials.
MCC9-12.A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Use complex numbers in polynomial identities and equations.
MCC9-12.N.CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Understand the relationship between zeros and factors of polynomials.
MCC9-12.A.APR.2 Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).
MCC9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Solving polynomials that have a degree greater than those solved in Coordinate Algebra and Analytic Geometry is going to require the use of skills that were developed when we solved quadratics last year. Let’s begin by taking a look at some second degree polynomials and the strategies used to solve them. These equations have the form \( ax^2 + bx + c = 0 \), and when they are graphed the result is a parabola.

1. Factoring is used to solve quadratics of the form \( ax^2 + bx + c = 0 \) when the roots are rational. Find the roots of the following quadratic functions:

   a. \( f(x) = x^2 - 5x - 14 \)

   b. \( f(x) = x^2 - 64 \)

   c. \( f(x) = 6x^2 + 7x - 3 \)

   d. \( f(x) = 3x^2 + x - 2 \)
2. Another option for solving a quadratic whether it is factorable but particularly when it is not is to use the quadratic formula. Remember, a quadratic equation written in $ax^2 + bx + c = 0$ has solution(s)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Also remember that $b^2 - 4ac$ is the discriminant and gives us the ability to determine the nature of the roots.

- $b^2 - 4ac > 0$ \quad 2 \text{ real roots}
- $b^2 - 4ac = 0$ \quad 1 \text{ real root}
- $b^2 - 4ac < 0$ \quad 0 \text{ real roots (imaginary)}

Find the roots for each of the following. Also, describe the number and nature of these roots.

a. $f(x) = 4x^2 - 2x + 9$

b. $f(x) = 3x^2 + 4x - 8$

c. $f(x) = x^2 - 5x + 9$

3. Let’s take a look at the situation of a polynomial that is one degree greater. When the polynomial is a third degree, will there be any similarities when we solve?

Suppose we want to find the roots of $f(x) = x^3 + 2x^2 - 5x - 6$. By inspecting the graph of the function, we can see that one of the roots is distinctively 2. Since we know that $x = 2$ is a solution to $f(x)$, we also know that $x - 2$ is a factor of the expression $x^3 + 2x^2 - 5x - 6$. This means that if we divide $x^3 + 2x^2 - 5x - 6$ by $x - 2$ there will be a remainder of zero. Let’s confirm this with synthetic substitution:

\[
\begin{array}{c|cccc}
2 & 1 & 2 & -5 & -6 \\
& & 2 & 8 & 6 \\
\hline
1 & 4 & 3 & 0 \\
\end{array}
\]

Let’s practice synthetic division before we tackle how to solve cubic polynomials in general. Do the following division problems synthetically.

a. $\frac{10x^3 - 17x^2 - 7x + 2}{x - 2}$

b. $\frac{x^3 + 3x^2 - 10x - 24}{x + 4}$

c. $\frac{x^3 - 7x - 6}{x + 1}$
The main thing to notice about solving cubic polynomials (or higher degree polynomials) is that a polynomial that is divisible by \( x - k \) has a root at \( k \). Synthetic division applied to a polynomial and a factor result in a zero for the remainder. This leads us to the Factor Theorem, which states: A polynomial \( f(x) \) has a factor \( x - k \) if and only if \( f(k) = 0 \).

Solving cubic polynomials can be tricky business sometimes. A graphing utility can be a helpful tool to identify some roots, but in general there is no simple formula for solving cubic polynomials like the quadratic formula aids us in solving quadratics.

There is however a tool that we can use for helping us to identify Rational Roots of the polynomial in question.

4. The Rational Root Theorem states that any rational solutions to a polynomial will be in the form of \( \frac{p}{q} \) where \( p \) is a factor of the constant term of the polynomial (the term that does not show a variable) and \( q \) is a factor of the leading coefficient. This is actually much simpler than it appears at first glance.

   a. Let us consider the polynomial \( f(x) = x^3 - 5x^2 - 4x + 20 \)

      Identify \( p \) (all the factors of the constant term 20) = ______________________________

      Identify \( q \) (all the factors of the leading coefficient 1) = ______________________________

      Identify all possible combinations of \( \frac{p}{q} \): ______________________________

If \( f(x) = x^3 - 5x^2 - 4x + 20 \) is going to factor, then one of these combinations is going to “work”, that is, the polynomial will divide evenly. So the best thing to do is employ a little trial and error. Let’s start with the smaller numbers, they will be easier to evaluate.

Substitute for \( x \): 1, -1, 2, -2, 4, -4 ... 20, -20.

Why would substituting these values in for \( x \) be a useful strategy?

Why do we not have to use synthetic division on every one?

Define what the Remainder Theorem states and how it helps us.
Hopefully, you did not get all the way to -20 before you found one that works. Actually, 2 should have worked. Once there is one value that works, we can go from there.

Use the factor \( (x - 2) \) to divide \( f(x) \). This should yield:

\[
f(x) = x^3 - 5x^2 - 4x + 20 = (x - 2)(x^2 - 3x - 10)
\]

By factoring the result we can find all the factors: \( f(x) = x^3 - 5x^2 - 4x + 20 = (x - 2)(x + 2)(x - 5) \)

Therefore the roots are 2, -2, and 5.

What could be done if this portion was not factorable?

5. Use the Quadratic Formula

For each of the following find each of the roots, classify them and show the factors.

a. \( f(x) = x^3 - 5x^2 - 4x + 20 \)

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: _________________________________

Roots and Classification

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b. $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

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c. $f(x) = 4x^3 - 7x + 3$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

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6. What happens when we come to a function that is a 4th degree?

Well, just like the cubic there is no formula to do the job for us, but by extending our strategies that we used on the third degree polynomials, we can tackle any quartic function.

1st Develop your possible roots using the \( \frac{p}{q} \) method.

2nd Use synthetic division with your possible roots to find an actual root. If you started with a 4th degree, that makes the dividend a cubic polynomial.

3rd Continue the synthetic division trial process with the resulting cubic. Don’t forget that roots can be used more than once.

4th Once you get to a quadratic, use factoring techniques or the quadratic formula to get to the other two roots.

For each of the following find each of the roots, classify them and show the factors.

a. \( f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8 \)

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: 

Roots and Classification

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b. \( f(x) = x^4 - 11x^3 - 13x^2 + 11x + 12 \)

Possible rational roots:
Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: _________________________________

Roots and Classification

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\[ f(x) = x^5 - 12x^4 + 49x^3 - 90x^2 + 76x - 24 \]

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: _________________________________

Roots and Classification

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d. \( f(x) = x^5 - 5x^4 + 8x^3 - 8x^2 + 16x - 16 \)

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ________________________________

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7. Let’s consider a scenario where the roots are imaginary.

Suppose that you were asked to find the roots of \( f(x) = x^4 - x^3 + 3x^2 - 4x - 4 \).

There are only 6 possible roots: \( \pm 1, \pm 2, \pm 4 \). In the light of this fact, let’s take a look at the graph of this function.

It should be apparent that none of these possible solutions are roots of the function. And without a little help at this point we are absolutely stuck. None of the strategies we have discussed so far help us at this point.

a. But consider that we are given that one of the roots of the function is \( 2i \). Because roots come in pairs (think for a minute about the quadratic formula); an additional root should be \(-2i \). So, let’s take these values and use them for synthetic division.
b. Though the values may not be very clean, this process should work just as it did earlier. Take a moment and apply what you have been doing to this function.

Complete Factorization: ________________________________

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