Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them

Xena, a first grader, determines the sum of 6 + 5 by saying almost inaudibly, "Six," and then, while surreptitiously extending five fingers under her desk one at a time, counting, "Seven, eight, nine, ten, *eleven*." Yolanda, a second grader, tackles 6 + 5 by mentally reasoning that if 5 + 5 is 10 and 6 is 1 more than 5, then 6 + 5 must be 1 more than 10, or *11*. Zenith, a third grader, immediately and reliably answers, "Six plus five is eleven."

The three approaches just described illustrate the three phases through which children typically progress in mastering the basic number combinations—the single-digit addition and multiplication combinations and their complementary subtraction and division combinations:

- Phase 1: Counting strategies—using object counting (e.g., with blocks, fingers, marks) or verbal counting to determine an answer
- Phase 2: Reasoning strategies—using known information (e.g., known facts and relationships) to logically determine (deduce) the answer of an unknown combination
- Phase 3: Mastery—efficient (fast and accurate) production of answers

Educators generally agree that children should master the basic number combinations—that is, should achieve phase 3 as stated above (e.g., NCTM 2000). For example, in *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford, and Findell 2001), the National Research Council (NRC) concluded that attaining computational fluency—the efficient, appropriate, and flexible application of single-digit and multidigit calculation skills—is an essential aspect of mathematical proficiency.

Considerable disagreement is found, however, about how basic number combinations are learned, the causes of learning difficulties, and how best to help children achieve mastery. Although exaggerated to illustrate the point, the vignettes that follow, all based on actual people and events, illustrate the *conventional wisdom* on these issues or its practical consequences. (The names have been changed to protect the long-suffering.) This view is then contrasted with a radically different view (the

By Arthur J. Baroody

Art Baroody, baroody@uiuc.edu, is a professor of curriculum and instruction in the College of Education at the University of Illinois in Champaign. He is interested in the development of number and arithmetic concepts and skills among young children and those with learning difficulties.

number-sense view), originally advanced by William Brownell (1935) but only recently supported by substantial research.

Vignette 1: Normal children can master the basic number combinations quickly; those who cannot are mentally impaired, lazy, or otherwise at fault.

Alan, a third grader, appeared at my office extremely anxious and cautious. His apprehension was not surprising, as he had just been classified as "learning disabled" and, at his worried parents' insistence, had come to see the "doctor whose specialty was learning problems." His mother had informed me that Alan's biggest problem was that he could not master the basic facts. After an initial discussion to help him feel comfortable, we played a series of mathematics games designed to create an enjoyable experience and to provide diagnostic information. The testing revealed, among other things, that Alan had mastered some of the single-digit multiplication combinations, namely, the $n \times 0$, $n \times 1$, $n \times 2$, *and* $n \times 5$ *combinations. Although not at the level expected by his teacher, his performance was not seriously abnormal. (Alan's class had spent just one day each on the 10-fact families* $n \times 0$ *to* $n \times 9$, *and the teacher had expected everyone to master all 100 basic multiplication combinations in this time. When it was pointed out that mastering such combinations typically takes children considerably more time than ten days, the teacher revised her approach, saying, "Well, then, we'll spend two days on the hard facts like the 9-fact family.")*

Vignette 2: Children are naturally unmindful of mathematics and need strong incentives to learn it.

Bridget's fourth-grade teacher was dismayed and frustrated that her new students had apparently forgotten most of the basic multiplication and division facts that they had studied the previous year. In an effort to motivate her students, Mrs. Burnside lit a blowtorch and said menacingly, "You will learn the basic multiplication and division facts, or you will get burned in my classroom!" Given the prop, Bridget and her classmates took the threat literally, not figuratively, as presumably the teacher intended.

Vignette 3: Informal strategies are bad habits that interfere with achieving mastery and must be prevented or overcome.

Carol, a second grader, consistently won the weekly 'Round the World game. (The game entails *having the students stand and form a line and then asking each student in turn a question—for example, "What is 8 + 5?" or "How much is 9 take away 4?" Participants sit down if they respond incorrectly or too slowly, until only one child—the winner—is left.) Hoping to instruct and motivate the other students, Carol's teacher asked her, "What is your secret to winning? How do you respond accurately so quickly?" Carol responded honestly, "I can count really fast." Disappointed and dismayed by this response, the teacher wrote a note to Carol's parents explaining that the girl insisted on using "immature strategies" and was proud of it. After reading the note, Carol's mother was furious with her and demanded an explanation. The baffled girl responded, "But, Mom, I'm a really, really fast counter. I am so fast, no one can beat me."*

Vignette 4: Memorizing basic facts by rote through extensive drill and practice is the most efficient way to help children achieve mastery.

Darrell, a college senior, can still recall the answers to the first row of basic division combinations on a fifth-grade worksheet that he had to complete each day for a week until he could complete the whole worksheet in one minute. Unfortunately, he cannot recall what the combinations themselves were.

How Children Learn Basic Combinations

The conventional wisdom and the number-sense view differ dramatically about the role of phases 1 and 2 (counting and reasoning strategies) in achieving mastery and about the nature of phase 3 (mastery) itself.

Conventional wisdom: Mastery grows out of memorizing individual facts by rote through repeated practice and reinforcement.

Although many proponents of the conventional wisdom see little or no need for the counting and reasoning phases, other proponents of this perspective at least view these preliminary phases as opportunities to practice basic combinations or to imbue the basic combinations with meaning before they are memorized. Even so, all proponents of the conventional wisdom view agree that phases 1 and 2 are not *necessary* for achieving the storehouse of facts that is the basis of combination mastery. This conclusion is the logical consequence of the following common assumptions about mastering the number combinations and mental-arithmetic expertise:

- Learning a basic number combination is a simple process of forming an association or bond between an expression, such as 7 + 6 or "seven plus six," and its answer, 13 or "thirteen." This basic process requires neither conceptual understanding nor taking into account a child's developmental readiness—his or her existing everyday or informal knowledge. As the teachers in vignettes 1 and 4 assumed, forming a bond merely requires practice, a process that can be accomplished directly and in fairly short order without counting or reasoning, through flash-card drills and timed tests, for example.
- Children in general and those with learning difficulties in particular have little or no interest in learning mathematics. Therefore, teachers must overcome this reluctance either by profusely rewarding progress (e.g., with a sticker, smile, candy bar, extra playtime, or a good grade) or, if necessary, by resorting to punishment (e.g., a frown, extra work, reduced playtime, or a failing grade) or the threat of it (as the teacher in vignette 2 did).
- Mastery consists of a single process, namely, fact recall. (This assumption is made by the teacher and the mother in vignette 3.) Fact recall entails the automatic retrieval of the associated answer to an expression. This fact-retrieval component of the brain is independent of the conceptual and reasoning components of the brain.

Number-sense view: Mastery *that underlies computational fluency* grows out of discovering the numerous patterns and relationships that interconnect the basic combinations.

According to the number-sense view, phases 1 and 2 play an integral and necessary role in achieving phase 3; mastery of basic number combinations is viewed as an outgrowth or consequence of number sense, which is defined as well-interconnected knowledge about numbers and how they operate or interact. This perspective is based on the following assumptions for which research support is growing:

- Achieving mastery of the basic number combinations efficiently and in a manner that promotes computational fluency is probably more complicated than the simple associative-learning process suggested by conventional wisdom, for the reason that learning any large body of

Figure 1

Vertical keeping-track method (based on Wynroth [1986])

Phase 1. Encourage children to summarize the results of their informal multiplication computations in a table. For example, suppose that a child needs to multiply 7 × 7. She could hold up seven fingers, count the fingers once (to represent one group of seven), and record the result (7) on the first line below the ◯. She could repeat this process a second time (to represent two groups of seven) and write 14 on the next line. The child could continue this process until she has counted her fingers a seventh time to represent seven groups of seven, then record the answer, 49, on the seventh line below ◯. This written record could then be used later to compute the product of 8 × 7 (eight groups of seven). The child would just count down the list until she comes to the product for 7 × 7 (1 seven is 7, 2 sevens is 14, . . . , 7 sevens is 49) and count on seven more (50, 51, 52, 53, 54, 55, 56). In time, children will have created their own multiplication table.

×	①	②	③	④	⑤	⑥	⑦	⑧	⑨
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	
7	7	14	21	28	35	42	49	56	
8	8	16	24	32	40	48			
9	9	18	27		45				

Phase 2. Once the table is completed, children can be encouraged to find patterns or relationships within and between families. See, for example, part III ("Product Patterns") of probe 5.5 on page 5–25 and the "Multiplication and Division" section in box 5.6 on page 5–32, Baroody with Coslick (1998).

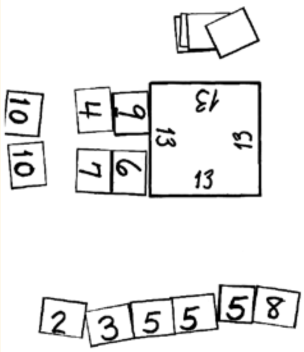
factual knowledge meaningfully is easier than learning it by rote. Consider, for example, the task of memorizing the eleven-digit number 25811141720. Memorizing this number by rote, even if done in chunks (e.g., 258-111-417-20), requires more time and effort than memorizing it in a meaningful manner—recognizing a pattern or relationship (start with 2 and repeatedly add 3). Put differently, psychologists have long known that people more easily learn a body of knowledge by focusing on its structure (i.e., underlying patterns and relationships) than by memorizing individual facts by rote. Furthermore, psychologists have long known that well-connected factual knowledge is easier to retain in memory and to transfer to learning other new but related facts than are isolated facts. As with any worthwhile knowledge, meaningful memorization of the basic combinations entails discovering patterns or relationships. For example, children who understand the "big idea" of *composition*—that a whole, such as a number, can be composed from its parts, often in different ways and with different parts (e.g., 1 + 7, 2 + 6, 3 + 5, and 4 + 4 = 8)—can recognize 1 + 7, 2 + 6, 3 + 5, and 4 + 4 as related facts, as a family of facts that "sum to eight." This recognition can help them understand the related big idea of *decomposition*—that a whole, such as a number, can be decomposed into its constitute parts, often in different ways (e.g., 8 = 1 + 7, 2 + 6, 3 + 5, 4 + 4, . . .). Children who understand the big ideas of composition and decomposition are more likely to invent reasoning strategies, such as translating combinations into easier or known expressions (e.g., 7 + 8 = 7 + [7 + 1] = [7 + 7] + 1 = 14 + 1 or 9 + 7 = 9 + [1 + 6] = [9 + 1] + 6 = 10 + 6 = 16). That is, children with a rich grasp of number and arithmetic patterns and relationships are more likely to achieve level 2.

- Children are intrinsically motivated to make sense of the world and, thus, look for regularities. Exploration and discovery are exciting to them.
- Combination mastery that ensures computational fluency may be more complicated than suggested by the conventional wisdom. Typi-

Figure 2

Examples of a composition-decomposition activity (based on Baroody, Lai, and Mix [in press])

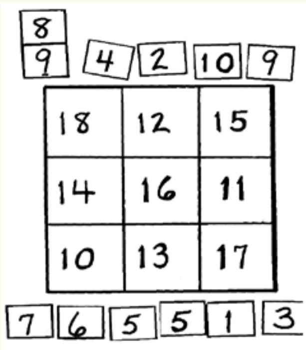
The Number Goal game



Two to six children can play this game. A large, square center card is placed in the middle with a number, such as 13, printed on it. From a pile of small squares, all facing down and having a number from 1 to 10, each player draws six squares. The players turn over their squares. Taking turns, each player tries to combine two or more of his or her squares to yield a sum equal to the number on the center card.

If a player had squares 2, 3, 5, 5, 5, and 8, she could combine 5 and 8 and also combine 3, 5, and 5 to make 13. Because each solution would be worth 1 point, the player would get 2 points for the round. If the player had chosen to combine 2 + 3 + 8, no other possible combinations of 13 would be left, and the player would have scored only 1 point for the round. An alternative way of playing (scoring) the game is to award points for both the number of parts used to compose the target number (e.g., the play 3 + 5 + 5 and 5 + 8 would be scored as 5 points, whereas the play 2 + 3 + 8 would be scored as 3 points).

Number Goal Tic-Tac-Toe (or Three in a Row)



This game is similar to the Number Goal game. Two children can play this game. From a pile of small squares, all facing down and having a number from 1 to 10, each player draws six squares. The players turn over their squares. Taking turns, each player tries to combine two or more of his or her squares to create a sum equal to one of the numbers in the 3 x 3 grid. If a player can do so, she or he places her or his marker on that sum in the 3 x 3 grid, discards the squares used, and draws replacement squares. The goal is the same as that for tic-tac-toe—that is, to get three markers in a row.

cally, with practice, many of the reasoning strategies devised in phase 2 become semiautomatic or automatic. Even adults use a variety of methods, including efficient reasoning strategies or—as Carol did in vignette 3—fast counting, to accurately and quickly determine answers to basic combinations. For example, children may first memorize by rote a few $n + 1$ combinations. However, once they realize that such combinations are related to their existing counting knowledge—specifically their already efficient number-after knowledge (e.g., "after 8 comes 9")—they do not have to repeatedly practice the remaining $n + 1$ combinations to produce them. That is, they discover the *number-after rule* for such combinations: "The sum of $n + 1$ is the number after n in the counting sequence." This reasoning process can be applied efficiently to any $n + 1$ combination for which a child knows the counting sequence, even those counting numbers that a child has not previously practiced, including large combinations, such as 1,000,128 + 1. (Note: The application of the number-after rule with multidigit numbers builds on previously learned and automatic rules for generating the counting sequence.) In time, the number-after rule for $n + 1$ combinations becomes automatic and can be applied quickly, efficiently, and without thought.

Recent research supports the view that the basic number-combination knowledge of mental-arithmetic experts is not merely a collection of isolated or discrete facts but rather a web of richly interconnected ideas. For example, evidence indicates not only that an understanding of commutativity enables children to learn all basic multiplication combinations by practicing only half of them but also that this conceptual knowledge may also enable a person's memory to store both combinations as a single representation. This view is supported by the observation that the calculation prowess of arithmetic savants does not stem from a rich store of isolated facts but from a rich number sense (Heavey 2003). In brief, phases 1 and 2 are essential for laying the conceptual groundwork—the discovery of patterns and relationships—and providing the reasoning strategies that underlie the attainment of computational fluency with the basic combinations in phase 3.

Reasons for Children's Difficulties

According to the conventional wisdom, learning difficulties are due largely to defects in the learner. According to the number-sense view, they are due largely to inadequate or inappropriate instruction.

Conventional wisdom: Difficulties are due to deficits inherent in the learner.

All too often, children's learning difficulties, such as Alan's as described in vignette 1, are attributed largely or solely to *their* cognitive limitations. Indeed, children labeled "learning disabled" are often characterized as inattentive, forgetful, prone to confusion, and unable to apply knowledge to even moderately new problems or tasks. As vignette 1 illustrates, these cognitive characteristics are presumed to be the result of mental-processing deficits and to account for the following nearly universal symptoms of children labeled learning disabled:

- A heavy reliance on counting strategies
- The capacity to learn reasoning strategies but an apparent inability to spontaneously invent such strategies
- An inability to learn or retain basic number combinations, particularly those involving numbers greater than 5 (e.g., sums over 10)
- A high error rate in recalling facts (e.g., "associative confusions," such as responding to 8 + 7 with "16"—the sum of 8 + 8—or with "56"—the product of 8 × 7)

In other words, children with learning difficulties, particularly those labeled learning disabled, seem to get stuck in phase 1 of number-combination development. They can sometimes achieve phase 2, at least temporarily, if they are taught reasoning strategies directly. Many, however, never achieve phase 3.

Number-sense view: Difficulties are due to defects inherent in conventional instruction.

Although some children labeled learning disabled certainly have impairments of cognitive processes, many or even most such children and other struggling students have difficulties mastering the basic combinations for two reasons. One is that, unlike their more successful peers, they lack adequate informal knowledge, which is a critical basis for understanding and successfully learning formal mathematics and devising effective problem-solving and reasoning strategies. For example, they may lack the informal experiences that allow them to construct a robust understanding of composition and decomposition; such understanding is foundational to developing many reasoning strategies.

A second reason is that the conventional approach makes learning the basic number combinations unduly difficult and anxiety provoking. The focus on memorizing individual combinations robs children of mathematical proficiency. For example, it discourages looking for patterns and relationships (conceptual learning), deflects efforts to reason out answers (strategic mathematical thinking), and undermines interest in mathematics and confidence in mathematical ability (a productive disposition). Indeed, such an approach even subverts computational fluency and creates the very symptoms of learning difficulties often attributed to children with learning disabilities and seen in other struggling children:

- *Inefficiency*. Because memorizing combinations by rote is far more challenging than meaningful memorization, many children give up on learning all the basic combinations; they may appear inattentive or unmotivated or otherwise fail to learn the combinations (as vignettes 1 and 4 illustrate). Because isolated facts are far more difficult to remember than interrelated ones, many children forget many facts (as vignette 2 illustrates). Put differently, as vignette 4 illustrates, a common consequence of memorizing basic combinations or other information by rote is forgetfulness. If they do not understand teacher-imposed rules, students may be prone to associative confusion. If a child does not understand why any number times 0 is 0 or why any number times 1 is the number itself, for instance, they may well confuse these rules with those for adding 0 and 1 (e.g., respond to 7 × 0 with "7" and to 7 × 1 with "8"). Because they are forced to rely on counting strategies and use these informal strategies surreptitiously and quickly, they are prone to errors (e.g., in an effort to use skip-counting by 7s to determine the product 4 × 7, or four groups of seven, a child might lose track of the number of groups counted, count "7, 14, 21," and respond "21" instead of "28").
- *Inappropriate applications*. When children focus on memorizing facts by rote instead of making sense of school mathematics or connecting it with their existing knowledge, they are more prone to misapply this knowledge because they make no effort to check themselves or they miss opportunities for applying what they do know (e.g., they fail to recognize that the answer "three" for 2 + 5 does not make sense). For example, Darrell's rote and unconnected knowledge in vignette 4 temporarily satisfied his teacher's demands but was virtually useless in the long run.
- *Inflexibility.* When instruction does not help or encourage children to construct concepts or look for patterns or relationships, they are less likely to spontaneously invent reasoning strategies, and thus they continue to rely on counting strategies. For example, children who do not have the opportunity to become familiar with composing and decomposing numbers up to 18 are unlikely to invent reasoning strategies for sums greater than 10.

Figure 3

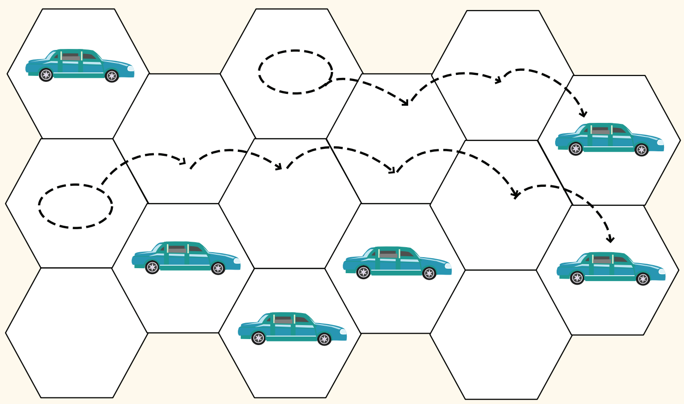
Road Hog car-race games

Additive composition version

In this version of Road Hog, each player has two race cars. The aim of the game is to be the first player to have both race cars reach the finish line. On a player's turn, he or she rolls two number cubes to determine how many spaces to move the car forward. (Cars may never move sideways or backward.) Play at the basic level involves two six-sided number cubes with 0 to 5 dots each. At the intermediate level, one number cube has 0 to 5 dots; the other, the numerals 0 to 5. This distinction may encourage counting on (e.g., for a roll of 4 and ⚂, a child might start with "four" and then count "five, six, seven" while successively pointing to the three dots). At the advanced level, both number cubes have the numerals 0 to 5. A similar progression of number cubes can be used for the super basic, super intermediate, and super advanced levels that involve sums up to 18 (i.e., played with six-sided number cubes, both having dots; or one number cube having dots and the other having numerals; or both number cubes having the numerals 5 to 9).

After rolling the number cubes, a player must decide whether to move each race car the number of spaces specified by one number cube in the pair (e.g., for a roll of 3 and 5, the player could move one car three spaces and the other five) or sum the two number cubes and move one car the distance specified by the sum (e.g., with a roll of 3 and 5, the player could move a single car the sum of 3 + 5, or 8, spaces). Note that in this version and all others, opponents must agree that the answer is correct. If an opponent catches the player in an error, the latter forfeits her or his turn.

Deciding which course to take depends on the circumstances of the game at the moment and a player's strategy. The racetrack, a portion of which is depicted below, consists of hexagons two or three wide. In the example depicted, by moving one car three spaces and the other car five spaces, the player could effectively block the road. The rules of the game specify that a car cannot go off the road or over another car. Thus, the cars of other players must stop at the roadblock created by the "road hog"—regardless of what number they roll.



Additive decomposition versions

The game has two decomposition versions. In both versions, the game can be played at three levels of difficulty. The basic level involves cards depicting whole numbers from 1 to 5; the intermediate level, 1 to 10; and the advanced level, 2 to 18.

In single additive decomposition, a child draws a card, for example, 3+?, which depicts a part (3), and a missing part (□), and a second card, which depicts the whole (5). The child must then determine the missing part (2), move one car a number of spaces equal to the known part (3), and move the other car a number of spaces equal to the missing part (2).

In double additive decomposition, a child draws a number card, such as 5, and can decompose it into parts any way she or he wishes (e.g., moving one car five spaces and the other none or moving one car three spaces and the other two).

Teachers may wish to tailor the game to children's individual needs. For example, for highly advanced children, the teacher may set up a desk with the version of the game involving whole numbers from 10 to 18.

Multiplicative decomposition versions

The multiplicative decomposition versions would be analogous to those for additive decomposition. For example, at the basic level, a player would draw a card—for example, 3 + ?—and would have to determine the missing factor—6. At the advanced level, a player would draw a card—for example, 18—and would have to determine both nonunit factors—2 and 9 or 3 and 6 (1 and 18 would be illegal).

Helping Children Master Basic Combinations

Proponents of the conventional wisdom recommend focusing on a short-term, direct approach, whereas those of the number-sense view recommend a long-term, indirect approach.

Conventional wisdom: Mastery can best be achieved by well-designed drill.

According to the conventional wisdom, the best approach for ensuring mastery of basic number combinations is extensive drill and practice. Because children labeled learning disabled are assumed to have learning or memory deficits, "over-learning" (i.e., massive practice) is often recommended so that such children retain these basic facts.

In recent years, some concern has arisen about the brute-force approach of requiring children, particularly those labeled learning disabled, to memorize all the basic combinations of an operation in relatively short order (e.g., Gersten and Chard [1999]). That is, concerns have been raised about the conventional approach of practicing and timed-testing many basic combinations at once. Some researchers have recommended limiting the number of combinations to be learned to a few at a time, ensuring that these are mastered before introducing a new set of combinations to be learned. A controlled- or constant-response-time procedure entails giving children only a few seconds to answer and providing them the correct answer if they either respond incorrectly or do not respond within the prescribed time frame. These procedures are recommended to minimize associative confusions during learning and to avoid reinforcing incorrect associations and "immature" (counting and reasoning) strategies. In this updated version of the conventional wisdom, then, phases 1 and 2 of number-combination development are still seen as largely unnecessary steps for, or even a barrier to, achieving phase 3.

Number-sense view: Mastery can best be achieved by purposeful, meaningful, inquiry-based instruction—instruction that promotes number sense.

A focus on promoting mastery of individual basic number combinations by rote does not make sense. Even if a teacher focuses on small groups of combinations at a time and uses other constant-response-time procedures, the limitations and difficulties of a rote approach largely remain. For this reason, the NRC recommends in *Adding It Up* that efforts to promote computational fluency be intertwined with efforts to foster conceptual understanding, strategic mathematical thinking (e.g., reasoning and problem-solving abilities), and a productive disposition. Four instructional implications of this recommendation and current research follow.

1. *Patiently help children construct number sense by encouraging them to invent, share, and refine informal strategies* (e.g., see phase 1 of **fig. 1**, p. 25). Keep in mind that number sense is not something that adults can easily impose. Help children gradually build up big ideas, such as composition and decomposition. (See **fig. 2**, p. 26, and **fig. 3**, p. 28, for examples of games involving these big ideas.) Children typically adopt more efficient strategies as their number sense expands or when they have a real need to do so (e.g., to determine an outcome of a dice roll in an interesting game, such as the additive composition version of Road Hog, described in **fig. 3**).

2. *Promote meaningful memorization or mastery of basic combinations by encouraging children to focus on looking for patterns and relationships;* to use these discoveries to construct reasoning strategies; and to share, justify, and discuss their strategies (see, e.g., phase 2 of **fig. 1**). Three major implications stem from this guideline:

   - Instruction should concentrate on "fact families," not individual facts, and how these combinations are related (see box 5.6 on pp. 5–31 to 5–33, Baroody with Coslick [1998] for a thorough discussion of the developmental bases and learning of these fact families).
   - Encourage children to build on what they already know. For example, mastering subtraction combinations is easier if children understand that such combinations are related to complementary and previously learned addition combinations (e.g., $5 - 3$ can be thought of as $3 + ? = 5$). Children who have already learned the addition doubles by discovering, for example, that their sums are the even numbers from 2 to 18, can use this existing knowledge to readily master $2 \times n$ combinations by recognizing that the latter is equal to the former (e.g., $2 \times 7 = 7 + 7 = 14$). Relating unknown combinations to previously learned ones can greatly reduce the amount of practice needed to master a family of combinations.
   - Different reasoning strategies may require different approaches. Research indicates that patterns and relationships differ in their salience. Unguided discovery learning might be appropriate for highly salient patterns or relationships, such as additive commutativity. More structured discovery learning activities may be needed for less obvious ones, such as the complementary relationships between addition and subtraction (see, e.g., **fig. 4**).

3. *Practice is important, but use it wisely.*

   - Use practice as an opportunity to discover patterns and relationships.
   - Practice should focus on making reasoning strategies more automatic, not on drilling isolated facts.
   - The learning and practice of number combinations should be done purposefully. Purposeful practice is more effective than drill and practice.

Figure 4

What's Related (based on Baroody [1989])

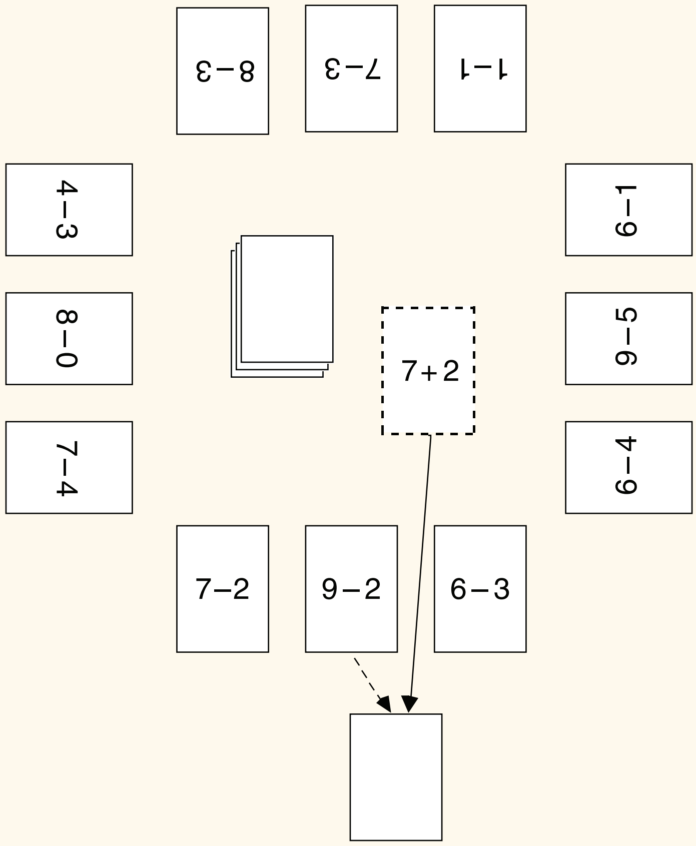
Objectives: (a) Reinforce explicitly the addition-subtraction complement principle and (b) provide purposeful practice of the basic subtraction combinations with single-digit minuends (basic version) or teen minuends (advanced version)

Grade level: 1 or 2 (basic version); 2 or 3 (advanced version)

Participants: Two to six players

Materials: Deck of subtraction combinations with single-digit minuends (basic version) or teen minuends (advanced version) and a deck of related addition combinations

Procedure: From the subtraction deck, the dealer deals out three cards faceup to each player (see figure). The dealer places the addition deck in the middle of the table and turns over the top card. The player to the dealer's left begins play. If the player has a card with a subtraction combination that is related to the combination on the addition card, he or she may take the cards and place them in a discard pile. The dealer then flips over the next card in the addition deck, and play continues. The first player(s) to match (discard) all three subtraction cards wins the game (short version) or a point (long version). (Unless the dealer is the first to go out, a round should be completed so that all players have an equal number of chances to make a match.)



- Practice to ensure that efficiency not be done prematurely—that is, before children have constructed a conceptual understanding of written arithmetic and had the chance to go through the counting and reasoning phases.

4. *Just as "experts" use a variety of strategies, including automatic or semi-automatic rules and reasoning processes, number-combination proficiency or mastery should be defined broadly as including any efficient strategy, not narrowly as fact retrieval.* Thus, students should be encouraged in, not discouraged from, flexibly using a variety of strategies.

Conclusion

An approach based on the conventional wisdom, including its modern hybrid (the constant-response-time procedure) can help children achieve mastery with the basic number combinations but often only with considerable effort and difficulty. Furthermore, such an approach may help children achieve efficiency but not other aspects of computational fluency—namely, appropriate and flexible application—or other aspects of mathematical proficiency—namely, conceptual understanding, strategic mathematical thinking, and a productive disposition. Indeed, an approach based on the conventional wisdom is likely to serve as a roadblock to mathematical proficiency (e.g., to create inflexibility and math anxiety).

Achieving computational fluency with the basic number combinations is more likely if teachers use the guidelines for meaningful, inquiry-based, and purposeful instruction discussed here. Children who learn the basic combinations in such a manner will have the ability to use this basic knowledge accurately and quickly (efficiently), thoughtfully in both familiar and unfamiliar situations (appropriately), and inventively in new situations (flexibly). Using the guidelines for meaningful, inquiry-based, and purposeful approach can also help students achieve the other aspects of mathematical proficiency: conceptual understanding, strategic mathematical thinking, and a productive disposition toward learning and using mathematics. Such an approach can help all children and may be particularly helpful for children who have been labeled learning disabled but who do not exhibit hard signs of cognitive dysfunction. Indeed, it may also help those with genuine genetic or acquired disabilities.

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